18.152 PROBLEM SET 1

due February 12th 9:30 am

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. *Let $u(x,t)$ be the smooth solution to the following Cauchy-Neumann problem;*

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \qquad \text{for } 0 \le x \le L, 0 \le t, \\ &u_x(0,t) = u_x(L,t) = 0, \qquad \text{for } 0 \le t, \\ &u(x,0) = g(x), \qquad \text{for } 0 \le x \le L, \end{aligned}$$

where $g(x)$ is smooth. Show the following inequality

$$\frac{d}{dt}\int_0^L |u_x(x,t)|^2 + |u_t(x,t)|^2 dx \le 0.$$

Problem 2. *Let $u(x,t)$ be the smooth solution to the following Cauchy-Neumann problem;*

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \qquad \text{for } 0 \le x \le 1, 0 \le t, \\ &u_x(0,t) = u_x(1,t) = 0, \qquad \text{for } 0 \le t, \\ &u(x,0) = g(x), \qquad \text{for } 0 \le x \le 1, \end{aligned}$$

where $g(x)$ is smooth. Show that $u(x,t)$ uniformly converges to the constant $\int_0^1 g(s)ds$ as $t \to +\infty$ by using the following steps.

(1) *Show that*

$$\int_0^1 |u_x(x,t)|^2 dx \le e^{-\frac{t}{2}} \int_0^1 |g'(x)|^2 dx$$

(2) *Show that*

$$\left| u(x,t) - \int_0^1 g(x)dx \right|^2 \le 4e^{-\frac{t}{2}} \int_0^1 |g'(x)|^2 dx.$$

Hint: Use Lemma 3 and Theorem 4 in lecture notes.

Problem 3. *Show that the following Cauchy-Dirichlet problem has a unique smooth solution, and express the solution in exact form.*

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \qquad \text{for } 0 \le x \le \pi, 0 \le t, \\ &u(0,t) = 0, u(\pi,t) = 2\pi, \qquad \text{for } 0 \le t, \\ &u(x,0) = 2x + \sin x + \sin(2x), \qquad \text{for } 0 \le x \le \pi. \end{aligned}$$

Hint: Consider $v(x,t) = u(x,t) - 2x$.

Problem 4. *Let $u(x,t)$ be the smooth solution to the following Cauchy-Neumann problem;*

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \qquad \text{for } 0 \le x \le L, 0 \le t, \\ &u_x(0,t) = u_x(L,t) = 0, \qquad \text{for } 0 \le t, \\ &u(x,0) = g(x), \qquad \text{for } 0 \le x \le L, \end{aligned}$$

where $g(x)$ is smooth.

(1) *Show that*
$$|u(x,t)| \le \max_{0 \le x \le L} |g(x,0)|.$$
(2) *Show that the smooth function*
$$w = \tfrac{t}{t+1}|u_x|^2 + \tfrac{1}{2}u^2$$
satisfies
$$w_t \le w_{xx}$$
for all $(x,t) \in [0,L] \times [0,+\infty)$.
(3) *Establish an upper bound for $|u_x(x,t)|^2$ where $t > 0$ in terms of g and t.*